Introduction to Rational Expressions

Definition A ***rational expression*** is a fraction whose numerator and denominator are each a polynomial.

Remark 1 A rational expression is formed when one polynomial is divided by another. Here are some examples of rational expressions to help you wrap your head around the concept.

ex. $\longrightarrow$ $\dfrac{x^2+2x+1}{2x+5}$, $\dfrac{x+y}{x^2y+4xy^2}$, $\dfrac{5}{2x-3}$, $\dfrac{x}{5}$, $\dfrac{1}{x}$, $\dfrac{2}{3}$

Rational expressions can have several variables!

Regular fractions are also rational expressions because integers are constant polynomials!

Remark 2 Rational expressions are essentially fractions that happen to contain variables.† So, ***all the principles that apply to fractions can be extended to apply to rational expressions!***

In this module and the next we study how to extend fraction principles to deal with variables in the our fractions.

Remark 3 Any expression that has a variable in it, like the above rational expressions, takes on different values as the variable changes. Determining the value of an expression at a particular value of the variable is called ***evaluating*** the expression.

ex. $\longrightarrow$ Evaluate $\dfrac{5-x}{2x+4}$ at $x=1$.

To evaluate the expression at $x=1$ we must substitute in the value of 1 everywhere we see x.

$$\frac{5-x}{2x+4} \xrightarrow[x=1]{\text{Plug in}} \frac{5-(1)}{2(1)+4} = \frac{4}{6} = \boxed{\frac{2}{3}}$$

This tells us that when $x=1$ the value of $\dfrac{5-x}{2x+4}$ is $\dfrac{2}{3}$.

† Technically speaking the term "rational expression" is more specific than "a fraction that contains variables," but for now those technicalities are not super important. Keeping that in mind, it is encouraged to think of rational expressions as fractions with variables.

Remark 4 Unfortunately, evaluating rational expressions can sometimes give us trouble!

$\xrightarrow{\text{ex.}}$ Evaluate $\dfrac{5-x}{2x+4}$ at $x=-2$.

$$\frac{5-x}{2x+4} \xrightarrow[x=-2]{\text{Plug in}} \frac{5-(-2)}{2(-2)+4} = \frac{7}{0} \text{ !!!}$$

We've run into a big problem! The fraction $7/0$ is not defined! For this reason we would say that the rational expression $\frac{5-x}{2x+4}$ is undefined at $x=-2$.

Definition If a value of the variable makes the denominator of a rational expression zero, then the rational expression is said to be ***undefined*** at that value of the variable.

Remark 5 Many times it is useful to be able to determine where a rational expression is undefined.

$\xrightarrow{\text{ex.}}$ Determine the value of x that makes $\dfrac{2x+3}{5x-15}$ undefined.

Rational expressions are undefined when the denominator evaluates to zero. So, this question is asking us to find which value of x makes the denominator zero. To discover this we will set the denominator to zero and solve.

$$\frac{2x+3}{5x-15} \text{ is undefined when: } \underbrace{5x-15=0}_{\text{Set the denominator to 0}} \to 5x=15 \to \boxed{x=3}$$

This is a **"BAD"** x-value because it makes the original fraction undefined!

Main Idea

- ***To evaluate a rational expression:***
  Substitute the given value into the expression everywhere you see the variable.
- ***To find where a rational expression is undefined:***
  Set the denominator to zero and solve.
  - → Values determined in this manner are considered "bad" values. They are the only values of the variable for which the rational expression is undefined.

Example 1 Evaluate $\dfrac{x^2 - 16}{2x + 7}$ at the given values of x.

a $x = 1$

b $x = -5$

c $x = 4$

Example 2 Find all values of the variable that make each rational expression undefined.

a $\dfrac{x + 4}{x - 3}$

b $\dfrac{x + 4}{-3x}$

c $\dfrac{7}{w^2 - 4w - 12}$

d $\dfrac{2z}{z^2 + 1}$